

Oxford Physics Admissions Test Solutions

Oxford PAT paper 2021 Solutions

These solutions are provided by Dr Leonardo.
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1. What is the next number in the sequence? 1, 32, 243, 1024, 3125

[2]

A	B	C	D	E
5040	6225	7164	7776	8192

$$1 = 1^5$$

$$32 = 2^5$$

$$243 = 3^5$$

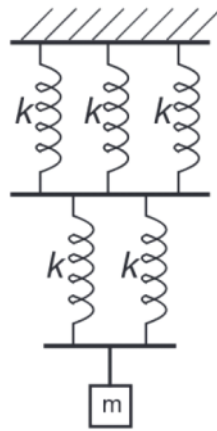
$$1024 = 4^5$$

$$3125 = 5^5$$

$$X = 6^5 = 7776$$

2. What is the effective spring constant of the combination of springs shown in the diagram, if each spring has spring constant k ?

[2]



A	B	C	D	E
$\frac{5}{6}k$	k	$\frac{6}{5}k$	$2k$	$5k$

STRINGS IN PARALLEL:

$$k = k_1 + k_2$$

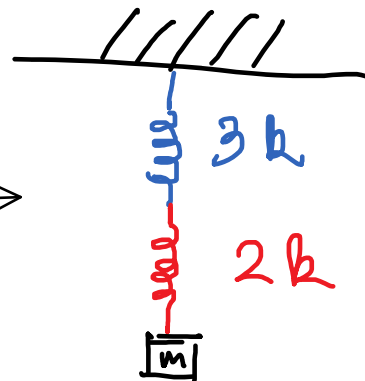
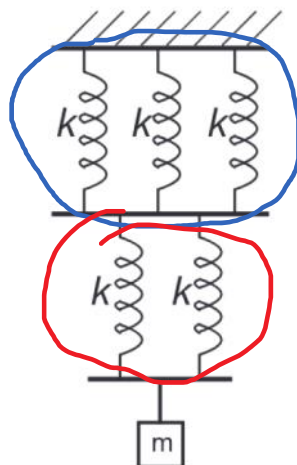
STRINGS IN SERIES:

$$\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2}$$

IN PARALLEL
 $k + k + k = 3k$

IN PARALLEL

$$k + k = 2k$$



SIMPLIFIED
 AS TWO
 STRINGS
 IN SERIES

$$\frac{1}{k_{\text{EFF}}} = \frac{1}{3k} + \frac{1}{2k} = \frac{2}{6k} + \frac{3}{6k} = \frac{5}{6k}$$

$$k_{\text{EFF}} = \frac{6}{5}k$$

3. Evaluate $\sum_{n=1}^{10} (2 - \frac{n}{2} + 2^n)$.

[2]

A	B	C	D	E
$2^{10} - \frac{11}{2}$	$2^{12} - \frac{19}{2}$	$2^{11} - \frac{19}{2}$	$2^{10} - \frac{11}{4}$	$2^{11} - \frac{11}{2}$

$$\sum_{n=1}^{10} (2 - \frac{n}{2} + 2^n) = \sum_{n=1}^{10} 2 - \frac{1}{2} \cdot \sum_{n=1}^{10} n + \sum_{n=1}^{10} 2^n$$

ADD 2 FOR EACH n

$$\sum_{n=1}^{10} 2 = 2 + 2 + \dots + 2 = 10 \cdot 2 = 20$$

ARITHMETIC SUM

$$S_N = a_1 + a_2 + \dots + a_N = \frac{N}{2} (a_1 + a_N)$$

$$\sum_{n=1}^{10} n = 1 + 2 + \dots + 10 = \frac{10}{2} \cdot (1 + 10) = 55$$

GEOMETRIC SUM

$$S_N = a_1 + a_1 r + a_1 r^2 + \dots + a_1 r^{N-1} = a_1 \frac{1-r^N}{1-r}$$

$$\sum_{n=1}^{10} 2^n = 2^1 + 2^2 + \dots + 2^{10} = 2 + 2 \cdot 2^1 + \dots + 2 \cdot 2^9 = 2 \cdot \frac{1-2^{10}}{1-2} = -2 + 2^{11}$$

$$\begin{aligned} \text{FINAL} &= 20 - \frac{1}{2} \cdot 55 + (-2 + 2^{11}) \\ &= -\frac{19}{2} + 2^{11} \end{aligned}$$

4. Five different ions are accelerated from rest by the same potential difference. Which will have the smallest final velocity?

[2]

A	B	C	D	E
${}^6_3\text{Li}^{2+}$	${}^7_3\text{Li}^{2+}$	${}^7_3\text{Li}^{3+}$	${}^9_4\text{Be}^{3+}$	${}^9_4\text{Be}^{4+}$

POTENTIAL EL. ENERGY $\Rightarrow$ KINETIC ENERGY

$$QV = \frac{1}{2}mv^2$$

$$v = \sqrt{\frac{2QV}{m}}$$



$\rightarrow$ ATOMIC MASS = $6m_u$
(m_u = UNIFIED ATOMIC MASS UNIT)

$$v_A = \sqrt{\frac{2 \cdot 2eV}{6m_u}} = \sqrt{\frac{2eV}{3m_u}}$$

$$v_B = \sqrt{\frac{2 \cdot 2eV}{7m_u}} = \sqrt{\frac{4eV}{7m_u}} \rightarrow \text{SMALLEST}$$

$$v_C = \sqrt{\frac{2 \cdot 3eV}{7m_u}} = \sqrt{\frac{6eV}{7m_u}}$$

$$v_D = \sqrt{\frac{2 \cdot 3eV}{9m_u}} = \sqrt{\frac{2eV}{3m_u}}$$

$$v_E = \sqrt{\frac{2 \cdot 4eV}{9m_u}} = \sqrt{\frac{8eV}{9m_u}}$$

5. Gravity on the Moon satisfies

$$g_{\text{Moon}} = \frac{1}{6}g_{\text{Earth}}.$$

A ball dropped on Earth from a height h takes a time t to reach the ground. From which height should it be dropped on the Moon so that it takes the same time t to reach the surface? You can neglect all effects of air resistance.

[2]

A	B	C	D	E
$\frac{1}{36}h$	$\frac{1}{6}h$	$\frac{1}{\sqrt{6}}h$	h	$6h$

$$s = ut + \frac{1}{2}at^2$$

$$\text{SINCE } u=0 \rightarrow s = \frac{1}{2}at^2$$

$$t = \sqrt{\frac{2s}{a}}$$

$$t_{\text{MOON}} = t_{\text{EARTH}}$$

$$\sqrt{\frac{2s_{\text{MOON}}}{a_{\text{MOON}}}} = \sqrt{\frac{2s_{\text{EARTH}}}{a_{\text{EARTH}}}} \quad /^2$$

$$\frac{s_{\text{MOON}}}{a_{\text{MOON}}} = \frac{s_{\text{EARTH}}}{a_{\text{EARTH}}}$$

$$\frac{s_{\text{MOON}}}{\frac{1}{6}g} = \frac{h}{g}$$

$$6s_{\text{MOON}} = h$$

$$s_{\text{MOON}} = \frac{h}{6}$$

6. Two unbiased dice are rolled and the numbers obtained are added. If the probability of getting the sum S is $P(S)$, which of the following statements are true?

[2]

1. $P(10) + P(11) = P(6)$
2. $P(6) > P(8)$
3. $P(2) + P(3) + P(4) > P(7)$
4. $P(7) = \frac{3}{2}P(5)$
5. $P(11) = P(3)$

A	B	C	D	E
1,2,4	3,4,5	2,3,4	1,3,5	1,4,5

$$P(x) = \frac{\text{NO. OF POSSIBILITIES THAT GIVE } x}{\text{NO. OF TOTAL POSSIBILITIES}}$$

NO. OF TOTAL POSSIBILITIES FOR 1 DICE = 6
FOR 2 DICE = $6 \cdot 6 = 36$

1) $10 = 6+4$ or $5+5$ or $4+6 \rightarrow 3$ POSSIBILITIES

$$P(10) = \frac{3}{36}$$

CAUTION! $4+6 = 1^{\text{st}}$ DICE 4 & 2^{nd} DICE 6
 $6+4 = 1^{\text{st}}$ DICE 6 & 2^{nd} DICE 4
NEED TO BE COUNTED SEPARATELY

$11 = 6+5$ or $5+6 \rightarrow 2$ poss.

$$P(11) = \frac{2}{36}$$

$6 = 1+5$ or $2+4$ or $3+3$ or $4+2$ or $5+1 \rightarrow 5$ poss.

$$P(6) = \frac{5}{36}$$

$$P(10) + P(11) = \frac{2}{36} + \frac{3}{36} = \frac{5}{36} = P(6) \text{ TRUE}$$

2) $8 = 6+2$ or $5+3$ or $4+4$ or $3+5$ or $6+2 \rightarrow 5$ poss

$$P(8) = \frac{5}{8} = P(6) \quad \text{FALSE}$$

3) $2 = 1+1 \rightarrow 1$ poss.

$$P(2) = \frac{1}{36}$$

$3 = 2+1$ or $1+2 \rightarrow 2$ poss.

$$P(3) = \frac{2}{36}$$

$4 = 3+1$ or $2+2$ or $1+3 \rightarrow 3$ poss.

$$P(4) = \frac{3}{36}$$

$7 = 6+1$ or $5+2$ or $4+3$ or $3+4$ or $2+5$ or $1+6 \rightarrow 6$ poss.

$$P(7) = \frac{6}{36}$$

$$P(2) + P(3) + P(4) = \frac{1}{36} + \frac{2}{36} + \frac{3}{36} = \frac{6}{36} = P(7) \quad \text{FALSE}$$

4) $5 = 4+1$ or $3+2$ or $2+3$ or $1+4 \rightarrow 4$ poss.

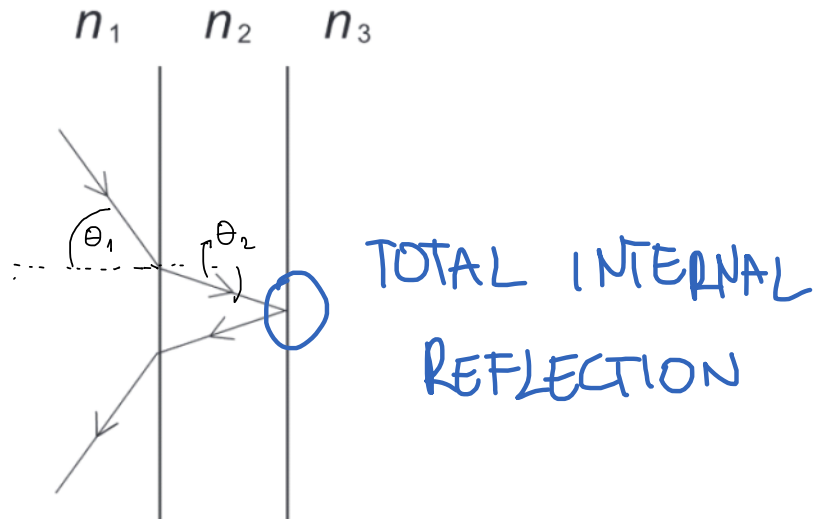
$$P(5) = \frac{4}{36}$$

$$\frac{3}{2} \cdot P(5) = \frac{6}{36} = P(7) \quad \text{TRUE}$$

$$5) \quad P(11) = P(3) = \frac{2}{36} \quad \text{TRUE}$$

7. A light ray follows a path through three media separated by plane boundaries as shown in the diagram, with refractive indices n_1 , n_2 and n_3 .

$$\theta_1 > \theta_2$$



Which of the following sequences puts the refractive indices in order of increasing value?

[2]

A	B	C	D	E
n_1, n_2, n_3	n_2, n_1, n_3	n_1, n_3, n_2	n_3, n_1, n_2	n_3, n_2, n_1

TOTAL INTERNAL REFLECTION ONLY HAPPENS WHEN LIGHT TRAVELS FROM HIGHER TO LOWER REFRACTIVE INDEX.

$$\rightarrow n_3 < n_2 \text{ AND } n_3 < n_1$$

n_1 & n_3 NOT IN DIRECT CONTACT, BUT IF THEY WERE, TOTAL REFLECTION WOULD STILL OCCUR.

SNELL'S LAW

$$\frac{n_1}{n_2} = \frac{\sin \theta_2}{\sin \theta_1}$$

SINCE $\theta_2 < \theta_1$

THEN $\sin \theta_2 < \sin \theta_1$

AND $n_1 < n_2$

$$\Rightarrow n_3 < n_1 < n_2$$

8. If $f(x) = x^2$ and $g(x) = x + 3$, find $\frac{dy}{dx}$ where $y = f(g(x)) - g(f(x))$.

[2]

A	B	C	D	E
6	$2x + 5$	$2x - 1$	$6x + 6$	2

$$f(g(x)) = f(x+3) = (x+3)^2$$

$$g(f(x)) = g(x^2) = x^2 + 3$$

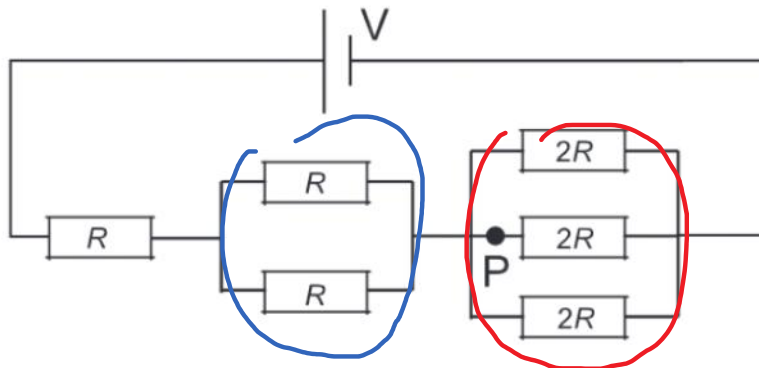
$$\begin{aligned} y &= (x+3)^2 - (x^2+3) \\ &= \cancel{x^2} + 6x + 9 - \cancel{x^2} - 3 \\ &= 6x + 6 \end{aligned}$$

$$\frac{dy}{dx} = 6$$

9. What is the current at the point P in the diagram?

[2]

A	B	C	D	E
$\frac{2V}{13R}$	$\frac{2V}{11R}$	$\frac{V}{9R}$	$\frac{6V}{13R}$	$\frac{6V}{11R}$



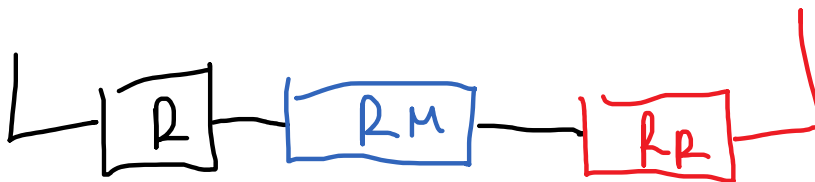
$$\frac{1}{R_M} = \frac{1}{R} + \frac{1}{R} = \frac{2}{R}$$

$$R_M = \frac{R}{2}$$

$$\frac{1}{R_R} = \frac{1}{2R} + \frac{1}{2R} + \frac{1}{2R} = \frac{3}{2R}$$

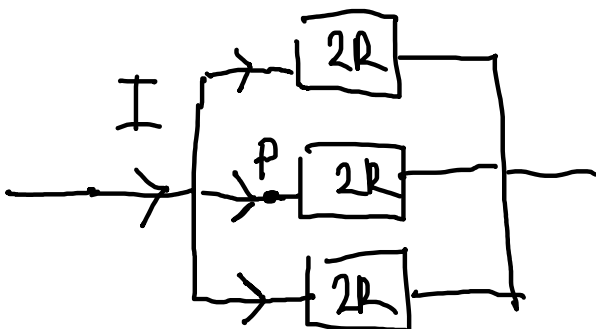
$$R_R = \frac{2R}{3}$$

→ EQUIVALENT CIRCUIT



$$R_{TOTAL} = R + R_M + R_R = R + \frac{1}{2}R + \frac{2}{3}R = \frac{13}{6}R$$

$$I = \frac{V}{R_{TOTAL}} = \frac{V}{\frac{13}{6}R} = \frac{6V}{13R}$$



I SPLITS INTO THREE
EQUAL CURRENTS

$$I_P = \frac{I}{3} = \frac{2V}{13R}$$

10. Which of these represents a simpler form for $\cos(\sin^{-1}(x))$?

[2]

A	B	C	D	E
$\sqrt{1-x^2}$	$\sqrt{1+x^2}$	$\sqrt{1-x}$	$1-x^2$	$\sqrt{x-1}$

RECALL $\cos^2 \alpha + \sin^2 \alpha = 1$

SO THAT $\cos \alpha = \sqrt{1 - \sin^2 \alpha}$

IF $\alpha = \sin^{-1} x$ THEN

$$\begin{aligned}\cos(\sin^{-1} x) &= \sqrt{1 - \sin^2(\sin^{-1} x)} \\ &= \sqrt{1 - (\sin(\sin^{-1} x))^2} \\ &= \sqrt{1 - x^2}\end{aligned}$$

BECAUSE $f^{-1}(f(x)) = x$

SO $\sin^{-1}(\sin x) = x$

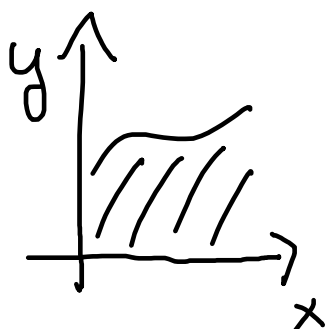
11. Consider the following five graphs.

1. Force (y -axis) against distance (x -axis)
2. Force (y -axis) against time (x -axis)
3. Velocity (y -axis) against time (x -axis)
4. Mass (y -axis) against velocity squared (x -axis)
5. Voltage (y -axis) against charge (x -axis)

For which graphs could the area under the graph potentially be a measurement of energy?

[2]

A	B	C	D	E
1, 4, 5	1, 5	1, 4	1, 3, 4	All of them.



$$\text{AREA} = x \cdot y$$

MORE PRECISELY

$$\text{AREA} = \int y \, dx$$

1. $F \cdot s = \text{WORK} = \text{ENERGY}$

2. $F \cdot t = \text{IMPULSE} \neq \text{ENERGY}$

3. $v \cdot t = \text{DISPLACEMENT} \neq \text{ENERGY}$

4. $m \cdot v = \text{MOMENTUM} \neq \text{ENERGY}$

5. $Q \cdot V = \text{POTENTIAL ELECTRICAL ENERGY}$

12. Which of the following integrals are equal to zero (you do **not** need to evaluate the integrals explicitly)?

1. $\int_{-\pi/2}^{\pi/2} \sin(3x) \, dx$

2. $\int_{-\sqrt{7}}^{\sqrt{7}} \frac{1}{9}x^5 - \frac{1}{7}x^3 + \frac{1}{21}x \, dx$

3. $\int_0^1 \cos x \, dx$

4. $\int_{-2}^2 \frac{1}{4}x^4 + \frac{1}{12}x^2 \, dx$

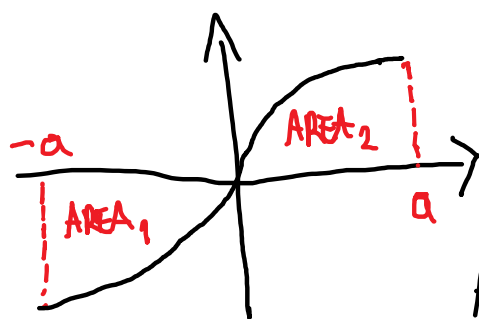
5. $\int_{-90}^{90} \sin(5x) - \frac{1}{2} \sin x \, dx$

A	B	C	<u>D</u>	E
1,2	3,4	1	1,2,5	1,5

ODD FUNCTION $\Rightarrow f(-x) = -f(x)$

ODD FUNCTIONS ARE ANTISYMMETRIC W.R.T y -AXIS

SO THEIR INTEGRAL FROM $-a$ to a IS ZERO.



$$\text{AREA}_1 = -\text{AREA}_2$$

$$\int_{-a}^a \text{ODD FUNCTION} = \text{AREA}_1 + \text{AREA}_2 = \text{AREA}_1 - \text{AREA}_1 = 0$$

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1.) $\sin(-3x) = -\sin(3x)$ ODD

so $\int_{-7}^7 \sin(3x) = 0$

$$\begin{aligned}
 2.) \quad \frac{1}{9}(-x)^5 - \frac{1}{7}(-x)^3 + \frac{1}{21}(-x) &= -\frac{1}{9}x^5 + \frac{1}{7}x^3 - \frac{1}{21}x \\
 &= -\left(\frac{1}{9}x^5 - \frac{1}{7}x^3 + \frac{1}{21}x\right) \\
 &\quad \text{ODD}
 \end{aligned}$$

$$\begin{aligned}
 3.) \quad \cos(-x) &= \cos(x) \quad \text{NOT ODD} \\
 \text{THIS IS EVEN} &\Rightarrow f(x) = f(-x)
 \end{aligned}$$

$$\begin{aligned}
 4.) \quad \frac{1}{4}(-x)^4 + \frac{1}{12}(-x)^2 &= \frac{1}{4}x^4 + \frac{1}{12}x^2 \\
 &\quad \text{NOT ODD BUT EVEN}
 \end{aligned}$$

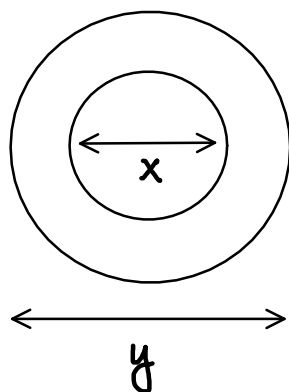
$$\begin{aligned}
 5.) \quad \sin(-5x) - \frac{1}{2}\sin(-x) &= -\sin 5x + \frac{1}{2}\sin x \\
 &= -\left(\sin 5x - \frac{1}{2}\sin x\right) \\
 &\quad \text{ODD}
 \end{aligned}$$

FUNCTIONS 1, 2, 5 ARE ODD

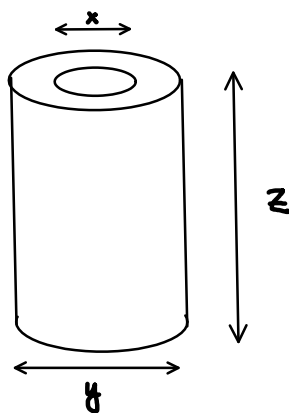
SO THEIR INTEGRALS ARE ZERO.

13. A toilet roll is made up of an inner cardboard tube, diameter x cm, with toilet paper wrapped around it to give an overall diameter of y cm. The length of the cardboard tube is z cm. Suppose the diameter of the inner tube is reduced from x cm to $(x - 1)$ cm, but the volume of toilet paper is kept the same. What is the difference in the total volume of the roll?

[3]



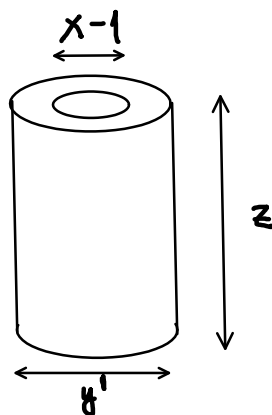
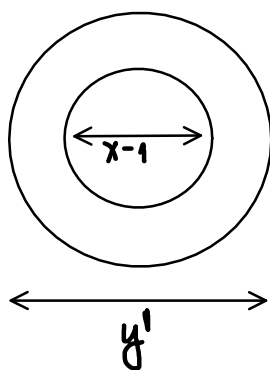
TOP VIEW



SIDE VIEW

$$V_{\text{PAPER}} = \pi (y^2 - x^2) z$$

$$V_{\text{ROLL}} = \pi y^2 \cdot z$$



$$V_{\text{PAPER}}' = \pi (y'^2 - (x-1)^2) z$$

$$V_{\text{ROLL}}' = \pi y'^2 z$$

$$= \pi (y^2 - 2x + 1) z$$

$$V_{\text{PAPER}}' = V_{\text{PAPER}}$$

$$\pi (y^2 - x^2) z = \pi (y'^2 - (x-1)^2) z$$

$$y^2 - x^2 = y'^2 - x^2 + 2x - 1$$

$$y'^2 = y^2 - 2x + 1$$

$$\Delta V_{\text{ROLL}} = V_{\text{ROLL}}' - V_{\text{ROLL}}$$

$$= \pi (y^2 - 2x + 1 - y^2) z$$

$$= \pi (1 - 2x) z$$

14. A machine requires 10kW of electrical power when operating at 100V.

- (a) If power is delivered through a cable with resistance 100Ω , how much power is lost to the cable's resistance? [2]
- (b) In order to reduce power loss, a transformer is used between the cable and the machine. What ratio of turns is required on the transformer in order to reduce power loss on the cable by a factor of 10^4 ? [2]

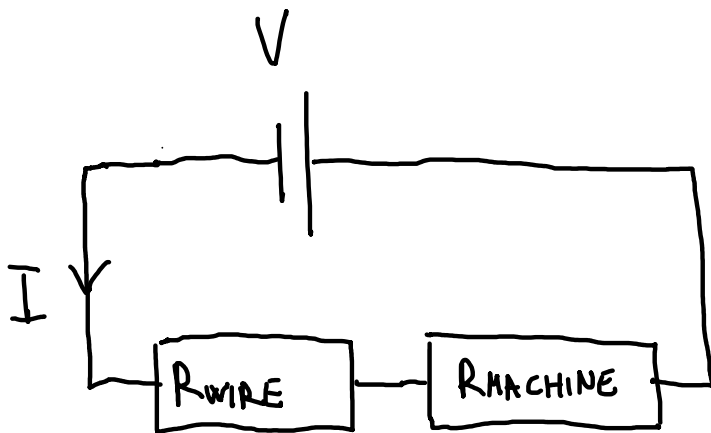
$$a) P = \frac{V^2}{R} \Rightarrow R = \frac{V^2}{P}$$

$$P_{\text{MACHINE}} = 10 \text{ kW} = 10\,000 \text{ W}$$

$$V_{\text{MACHINE}} = 100 \text{ V}$$

$$R_{\text{MACHINE}} = \frac{100^2}{10\,000} = 1 \Omega$$

REPRESENT THE WIRE AS ANOTHER RESISTOR IN SERIES



IN SERIES:

$$I = I_{\text{WIRE}} = I_{\text{MACHINE}}$$

$$I = \frac{V}{R} \text{ SO THAT}$$

$$\frac{V_{\text{WIRE}}}{R_{\text{WIRE}}} = \frac{V_{\text{MACHINE}}}{R_{\text{MACHINE}}}$$

$$\frac{V_{\text{WIRE}}}{100} = \frac{100}{1}$$

$$V_{\text{WIRE}} = 10\,000 \text{ V}$$

$$P_{\text{WIRE}} = \frac{V_{\text{WIRE}}^2}{R_{\text{WIRE}}} = \frac{10\,000^2}{100} = 10^6 \text{ W}$$

b) FOR A TRANSFORMER

$$\frac{N_1}{N_2} = \frac{V_1}{V_2}$$

$$P_2 = \frac{1}{10000} P_1$$

$$\frac{V_2^2}{R_{\text{WIRE}}} = \frac{1}{10000} \cdot \frac{V_1^2}{R_{\text{WIRE}}}$$

$$V_2^2 = \frac{1}{10000} V_1^2 \quad \sqrt{\quad}$$

$$V_2 = \frac{1}{100} V_1$$

$$\frac{V_1}{V_2} = \frac{N_1}{N_2} = 100$$

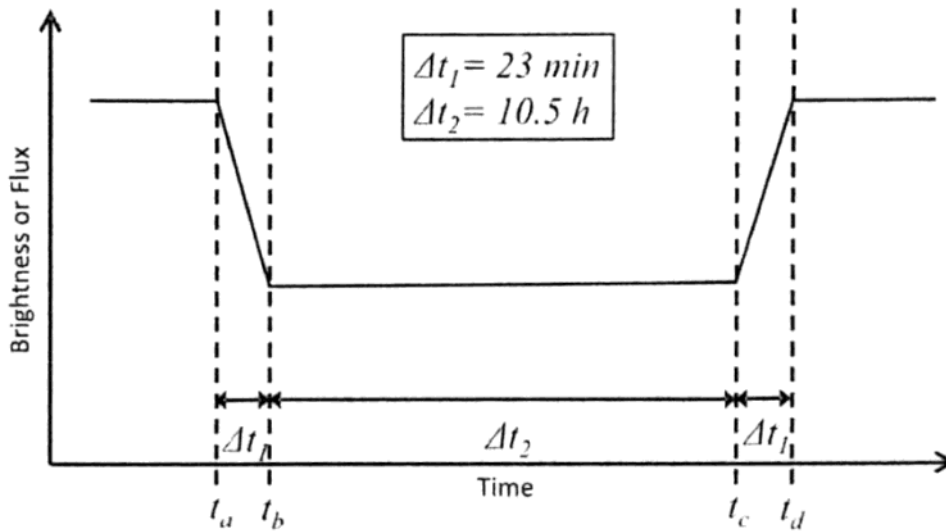
15. The observed brightness of a Sun-like star shows periodic dips. The time period between these dips $T = 225$ days.

- (a) If these dips are interpreted as being due to the transits of a planet in a circular orbit around the star, estimate the radius R of that orbit.

You may assume that the star has the same mass as the Sun and that the planet's mass is much smaller than the mass of the star. You can take the mean radius of the Earth's orbit around the Sun to be approximately $R_E = 1.5 \times 10^{11}$ m.

[5]

- (b) A model of a transit and its effect on the star's brightness is shown in the figure below. Assuming that we observe the system in the plane of the planet's orbit, draw the relative positions of the planet and the star at the times t_a to t_d and calculate the radii of the planet, R_P and the star, R_S .



[5]

Q) USE 3rd KEPLER'S LAW. LET'S DERIVE IT.

m = PLANET'S MASS

M = STAR'S MASS

$$m \cdot a_{\text{CENTRIPETAL}} = F_{\text{GRAVITY}}$$

$$m \frac{v^2}{r} = \frac{GMm}{r^2}$$

$$v^2 = \frac{GM}{r}$$

r = RADIUS OF ORBIT

v = ORBITING SPEED

THE PLANET MAKES A FULL CIRCLE IN ONE PERIOD

$$V = \frac{\text{FULL CIRCLE}}{\text{PERIOD}} = \frac{2\pi r}{T}$$

$$V^2 = \left(\frac{2\pi r}{T} \right)^2 = \frac{GM}{r}$$

$$\frac{4\pi^2 r^2}{T^2} = \frac{GM}{r}$$

$$\frac{r^3}{T^2} = \frac{GM}{4\pi^2}$$

CONSTANT SINCE
 $M_{\text{STAR}} = M_{\text{SUN}}$

WE HAVE JUST PROVED 3rd KEPLER'S LAW

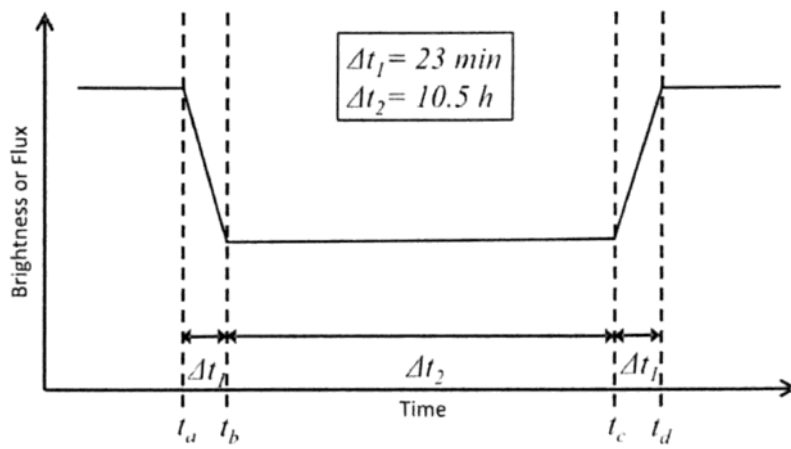
$$\frac{r^3}{T^2} = \text{const.}$$

$$\frac{r_{\text{PLANET}}^3}{T_{\text{PLANET}}^2} = \frac{r_E^3}{T_E^2}$$

$$\frac{r_{\text{PLANET}}^3}{225^2} = \frac{(1.5 \cdot 10^{11})^3}{365^2}$$

$$r_{\text{PLANET}} = 1.09 \cdot 10^{11} \text{ m}$$

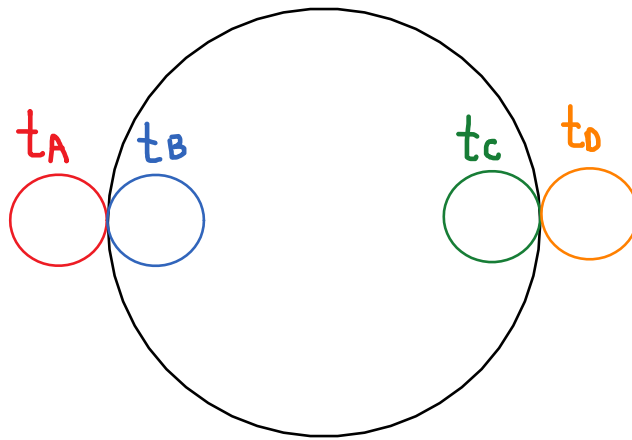
b)



[5]

AT t_A THE PLANET IS ABOUT TO START THE ECLIPSE; t_D IS THE OPPOSITE.
 AT t_B IT IS JUST FULLY IN FRONT OF THE STAR; t_C IS THE OPPOSITE.

STAR



THE LEFT EDGE OF THE PLANET TRAVELS:

i) DISTANCE $2R_p$ FROM t_A TO $t_B = \Delta t_1$

ii) DISTANCE $2R_s$ FROM t_B TO $t_D = \Delta t_1 + \Delta t_2$

$$V = \frac{2\pi r}{T} = \frac{2\pi \cdot 1.09 \cdot 10^{11}}{225 \cdot 24 \cdot 3600} = 35230 \text{ m/s}$$

$$2R_p = v \cdot \Delta t_1 \Rightarrow R_p = 2.43 \cdot 10^7 \text{ m}$$

$$2R_s = v \cdot (\Delta t_1 + \Delta t_2) \Rightarrow R_s = 6.9 \cdot 10^8 \text{ m}$$

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16. A vehicle travels in a fixed direction at a velocity $v(t)$ that varies with time as follows:

$$\begin{aligned} t < t_1 & \quad v(t) = At^2 \\ t_1 < t < t_2 & \quad v(t) = C - B(t - t_2)^2 \\ t > t_2 & \quad v(t) = v_2. \end{aligned}$$

Here A , B , C and v_2 are constants.

(a) Find A , B and C such that both the velocity and acceleration are continuous at $t = t_1$ and $t = t_2$. [3]

(b) Sketch the velocity and acceleration as a function of time. [3]

(c) Find the total distance travelled between $t = 0$ and $t = t_3$, where $t_3 > t_2$. [4]

a)

$$a(t) = \frac{dv}{dt} = \begin{aligned} & t < t_1 & a(t) &= 2At \\ & t_1 < t < t_2 & a(t) &= -2B(t - t_2) \\ & t > t_2 & a(t) &= 0 \end{aligned}$$

@ t_2 $v(t_2) = C - B(t_2 - t_2)^2 = v_2$
 $C = v_2$

@ t_1 $a(t_1) = 2At_1 = -2B(t_1 - t_2)$
 $At_1 = B(t_2 - t_1)$

$$v(t_1) = At_1^2 = C - B(t_1 - t_2)^2$$

$$At_1 \cdot t_1 = v_2 - B(t_1 - t_2)^2$$

$$B(t_2 - t_1) \cdot t_1 = v_2 - B(t_1^2 - 2t_1t_2 + t_2^2)$$

$$Bt_1t_2 - Bt_1^2 = v_2 - Bt_1^2 + 2Bt_1t_2 - Bt_2^2$$

$$Bt_2^2 - Bt_1t_2 = v_2$$

$$B = \frac{v_2}{t_2^2 - t_1t_2} = \frac{v_2}{t_2(t_2 - t_1)}$$

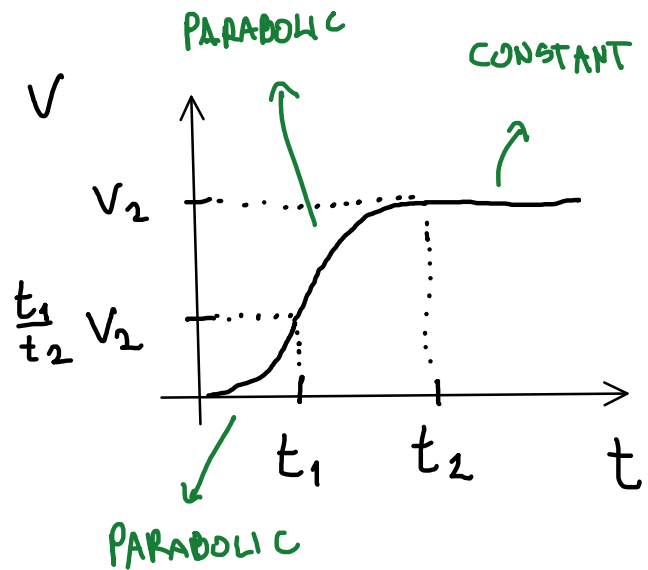
$$A t_1 = B (t_2 - t_1) = \frac{V_2}{t_2(t_2 - t_1)} \cdot (t_2 - t_1)$$

$$A = \frac{V_2}{t_1 t_2}$$

b)

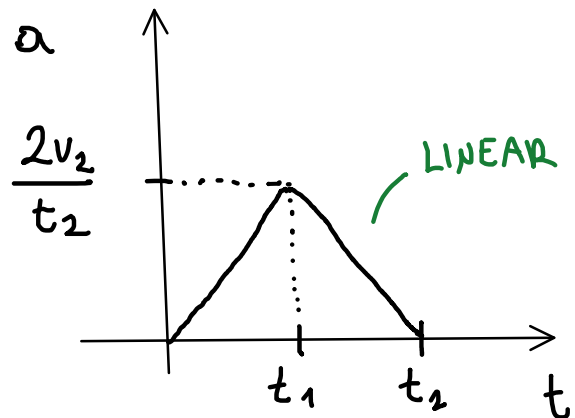
$$v(t_1) = \frac{V_2}{t_1 t_2} \cdot t_1^2 = V_2 \frac{t_1}{t_2}$$

$$v(t_2) = V_2$$



$$a(t_1) = 2 \cdot \frac{V_2}{t_1 t_2} \cdot t_1 = \frac{2V_2}{t_2}$$

$$a(t_2) = 0$$



$$c) s = \int v dt$$

$$= \int_0^{t_1} A t^2 dt + \int_{t_1}^{t_2} [C - B(t-t_2)^2] dt + \int_{t_2}^{t_3} v_2 dt$$

$$= \left[A \frac{t^3}{3} \right]_0^{t_1} + \left[C t - B \frac{(t-t_2)^3}{3} \right]_{t_1}^{t_2} + \left[v_2 t \right]_{t_2}^{t_3}$$

$$= \frac{A}{3} t_1^3 + C t_2 - B \frac{(t_2-t_2)^3}{3} - C t_1 + B \frac{(t_1-t_2)^3}{3} + v_2 (t_3 - t_2)$$

$$= \frac{v_2}{3 t_1 t_2} t_1^3 + \cancel{v_2 t_2} - v_2 t_1 + \frac{v_2}{t_2(t_2-t_1)} \cdot \frac{(t_1-t_2)^3}{3} + v_2 t_3 - \cancel{v_2 t_2}$$

$$= \frac{v_2 t_1^2}{3 t_2} - v_2 t_1 - \frac{v_2}{t_2(t_1-t_2)} \cdot \frac{(t_1-t_2)^3}{3} + v_2 t_3$$

$$= \frac{1}{3} v_2 \frac{t_1^2}{t_2} - v_2 t_1 - \frac{v_2}{3 t_2} \cdot (t_1^2 - 2 t_1 t_2 + t_2^2) + v_2 t_3$$

$$= \cancel{\frac{1}{3} v_2 \frac{t_1^2}{t_2}} - v_2 t_1 - \cancel{\frac{1}{3} v_2 \frac{t_1^2}{t_2}} + \frac{2}{3} v_2 t_1 - \frac{1}{3} v_2 t_2 + v_2 t_3$$

$$= -\frac{1}{3} v_2 t_1 - \frac{1}{3} v_2 t_2 + v_2 t_3 = v_2 \left(t_3 - \frac{1}{3} t_1 - \frac{1}{3} t_2 \right)$$

PHEW! IF YOU'VE FOLLOWED ALL THE WAY HERE,
REWARD YOURSELF WITH A COOKIE!

17. In Einstein's theory of gravity, light passing a star of mass M , at a distance R from the centre of the star, is bent by an angle θ (measured in radians, 2π radians = 360°) as shown in the diagram.

- (a) Assume that θ depends on the gravitational constant G and also depends on the mass of the star M , the distance R and the speed of light $c = 3 \times 10^8 \text{ m s}^{-1}$, as

$$\theta = \lambda G M^\alpha R^\beta c^\gamma,$$

where λ is an undetermined dimensionless constant.

By considering the dimensions of these quantities determine α , β and γ .

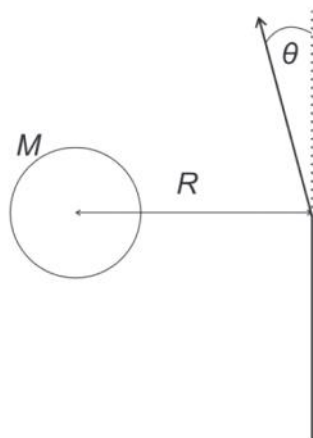
→ DIMENSIONAL ANALYSIS

[4]

- (b) In SI units the numerical value of G is 6.67×10^{-11} .

Neglecting any spatial extent to the star, and taking $\lambda = 1$ and $M = 2 \times 10^{30} \text{ kg}$, determine the distance R such that light is bent so much that it falls directly into the star.

[3]



a) WHAT ARE THE UNITS OF G ?

$$F = \frac{G M_1 M_2}{R^2}$$

$$[F] = N = \text{kg} \cdot \frac{\text{m}}{\text{s}^2}$$

$$G = \frac{F \cdot R^2}{M_1 M_2}$$

$$[G] = \frac{N \cdot \text{m}^2}{\text{kg}^2} = \frac{\text{kg} \cdot \frac{\text{m}}{\text{s}^2} \cdot \text{m}^2}{\text{kg}^2} = \frac{\text{m}^3}{\text{kg s}^2}$$

$$\theta = \lambda G M^\alpha R^\beta c^\gamma$$

$$[\theta] = \frac{\text{m}^3}{\text{kg s}^2} \cdot \text{kg}^\alpha \text{m}^\beta \left(\frac{\text{m}}{\text{s}}\right)^\gamma = \text{kg}^{\alpha-1} \text{s}^{-2-\gamma} \text{m}^{3+\beta+\gamma}$$

θ IS UNITLESS, SO $\alpha - 1 = 0 \Rightarrow \alpha = 1$

$$-2 - \gamma = 0 \Rightarrow \gamma = -2$$

$$3 + \beta - 2 = 0 \Rightarrow \beta = -1$$

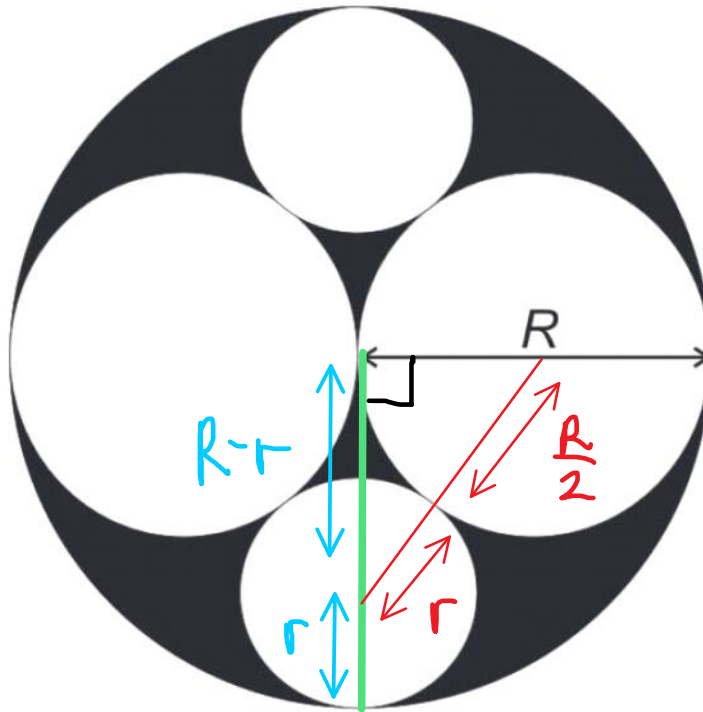
$$b) \quad \theta = \frac{\lambda GM}{Rc^2}$$

IF LIGHT FALLS DIRECTLY ONTO THE STAR,

$$\theta = 90^\circ = \frac{\pi}{2}$$

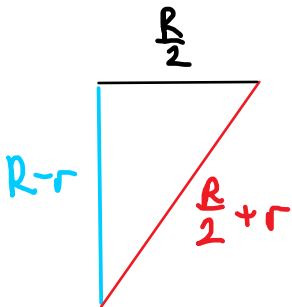
$$R = \frac{\lambda GM}{\theta c^2} = \frac{1 \cdot 6.67 \cdot 10^{-11} \cdot 2 \cdot 10^{30}}{\frac{\pi}{2} \cdot (3 \cdot 10^8)^2} = 943.6 \text{ m}$$

18. Find the area of the dark regions of this figure in terms of the overall radius of the outer circle (denoted R). [5]



GREEN LINE = R

NOTICE THE RIGHT-ANGLE TRIANGLE.



PYTHAGORA'S THEOREM:

$$\left(\frac{R}{2} + r\right)^2 = (R - r)^2 + \left(\frac{R}{2}\right)^2$$

$$\cancel{\frac{R^2}{4}} + \cancel{Rr} + \cancel{r^2} = \cancel{R^2} - \cancel{2Rr} + \cancel{r^2} + \cancel{\frac{R^2}{4}}$$

$$\cancel{3Rr} = \cancel{R^2} \Rightarrow r = \frac{R}{3}$$

$$\text{SHADED AREA} = \pi R^2 - 2 \cdot \pi \left(\frac{R}{2}\right)^2 - 2 \cdot \pi r^2$$

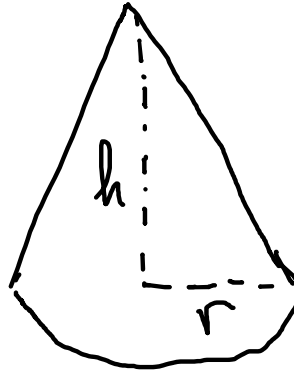
$$= \pi R^2 - \pi \frac{R^2}{2} - 2\pi \frac{R^2}{9}$$

$$= \frac{5\pi R^2}{18}$$

19. Sand is poured at a constant rate onto a flat horizontal surface. It forms a pile in the shape of a cone with a constant slope.

(a) If r is the radius of the base of the cone, find how r varies with t , given that $r(0) = 0$. [3]

(b) If it takes a time t_1 for r to reach r_1 , how long does it take for the radius to reach $2r_1$ (in terms of t_1)? [2]



$$\text{SLOPE} = k$$

SINCE THE SLOPE IS CONSTANT,

$$k = \frac{\text{RISE}}{\text{RUN}} = \frac{h}{r} \Rightarrow h = k \cdot r$$

$$V = \frac{1}{3} \pi r^2 h = \frac{1}{3} \pi r^2 \cdot k r = \frac{1}{3} \pi k r^3$$

$$\text{CONSTANT RATE OF POURING} = g = \frac{dV}{dt}$$

$$\Rightarrow V = g \cdot t = \frac{1}{3} \pi k r^3$$

$$r = \left(\frac{3gt}{\pi k} \right)^{\frac{1}{3}} \Rightarrow r \propto t^{\frac{1}{3}}$$

b) SINCE $r \propto t^{\frac{1}{3}}$

$$\frac{r_2}{r_1} = \left(\frac{t_2}{t_1} \right)^{\frac{1}{3}}$$

$$2 = \left(\frac{t_2}{t_1} \right)^{\frac{1}{3}} \Rightarrow \frac{t_2}{t_1} = 2^3 = 8$$

BINOMIAL EXPANSION

20. Expand $(a + bx)^c$, where c is a positive integer, as a polynomial in x up to and including terms involving x^2 .

If the coefficient of the x^0 term is equal to $\frac{1}{16}$, $b = \frac{1}{4}$ and the coefficient of the x^2 term is $\frac{3}{32}$, find a and c .

[4]

$$\text{REMINDER} = \binom{n}{m} = \frac{n!}{m!(n-m)!}$$

$$(a+bx)^c = \binom{c}{0} \cdot a^c + \binom{c}{1} a^{c-1} \cdot bx + \binom{c}{2} a^{c-2} \cdot (bx)^2 \dots$$

$$\binom{c}{0} = \frac{c!}{0!c!} = 1 \quad \text{CAUTION } 0! = 1$$

$$\binom{c}{1} = \frac{c!}{1!(c-1)!} = \frac{c \cdot (c-1)!}{(c-1)!} = c$$

$$\binom{c}{2} = \frac{c!}{2!(c-2)!} = \frac{c(c-1) \cdot (c-2)!}{2 \cdot (c-2)!} = \frac{c(c-1)}{2}$$

LET'S PLUG THESE BACK IN

$$(a+bx)^c = a^c + c a^{c-1} bx + \frac{c(c-1)}{2} a^{c-2} b^2 x^2$$

$$= a^c + \frac{1}{4} c a^{c-1} x + \frac{c(c-1)}{32} \frac{a^c}{a^2} x^2$$

$$a^c = \frac{1}{16}$$

$$\frac{c(c-1)}{32} \cdot \frac{1}{16} \cdot \frac{1}{a^2} = \frac{3}{32}$$

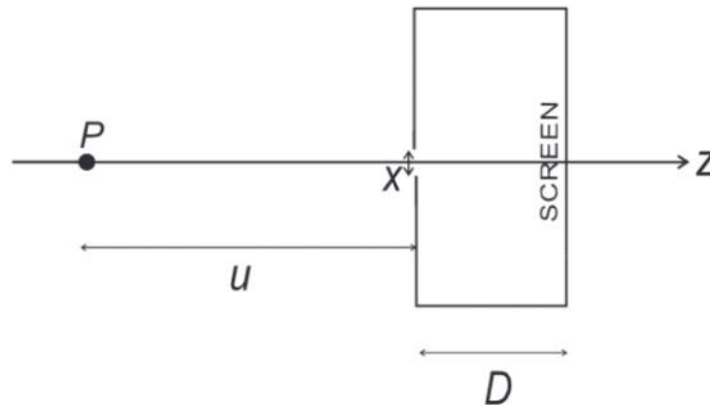
$$c(c-1) = 3 \cdot 16 \cdot a^2$$

↳ CAN ONLY BE CANCELED IF

$$c-1=3 \Rightarrow c=4$$

$$a = \frac{1}{2}$$

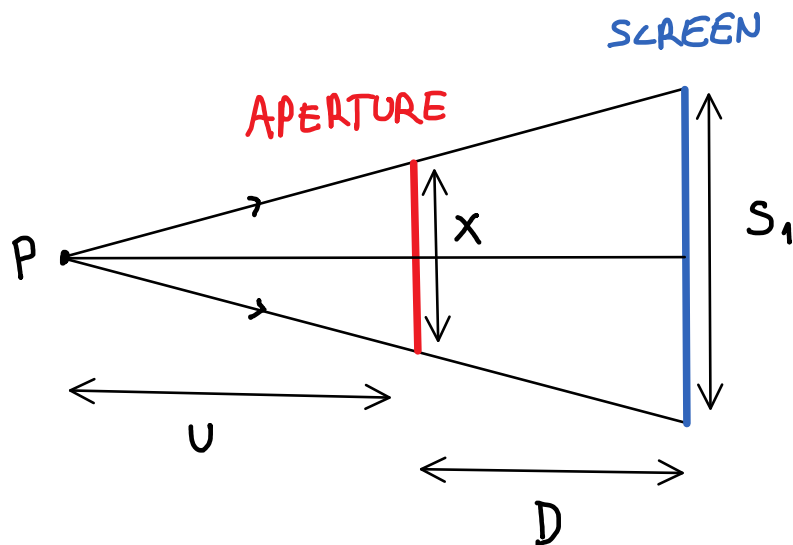
21. A pinhole camera consists of a box with a small aperture (of diameter x). A point light source P is at a distance u from the aperture outside the box. It projects an image on the screen, which is inside the box at a distance D from the aperture.



- (a) If diffraction is ignored, what is the diameter s_1 of the spot projected on the screen?

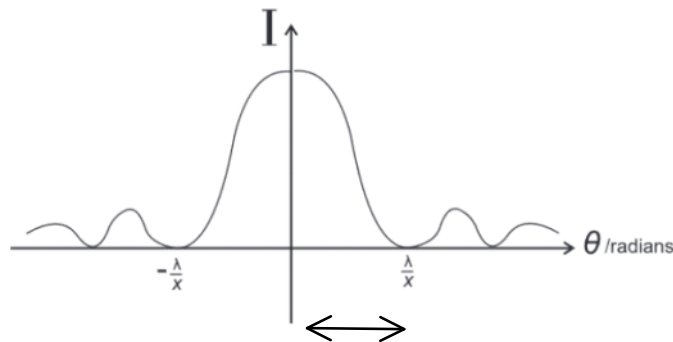
[2]

DRAW THE LIGHT RAYS AND USE TRIANGLE SIMILARITY:



$$\frac{s_1}{u+D} = \frac{x}{u} \Rightarrow s_1 = x \cdot \frac{u+D}{u}$$

- (b) If $D \gg x$, diffraction of light of wavelength λ by the aperture produces a distribution of intensity that varies with angle θ (measured in radians from the z axis marked on the diagram; 2π radians = 360 degrees) approximately as shown below:



$\theta_{\text{MIN}} = \text{POSITION OF FIRST MIN}$

What is the approximate diameter, s_2 , of the central peak of the diffraction pattern projected on the screen?

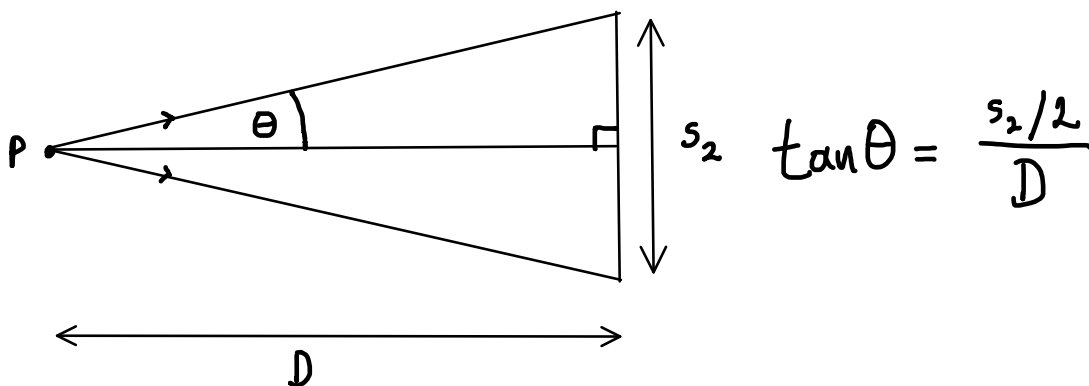
[2]

THIS IS SINGLE-SLIT DIFFRACTION
CONDITION FOR MINIMUM:

$$x \sin \theta = \lambda$$

CAUTION:

DO NOT CONFUSE WITH THE CONDITION
FOR MAX IN DOUBLE-SLIT DIFFRACTION



SMALL-ANGLE APPROXIMATION: $\theta \approx \tan \theta \approx \sin \theta$

$$x \cdot \sin \theta = x \cdot \frac{s_2}{2D} = \lambda$$

$$\Rightarrow s_2 = \frac{2\lambda D}{x}$$

- (c) Assume that the effects in part (a) and (b) can be combined to give a total diameter s_3 , where

$$s_3^2 = s_1^2 + s_2^2.$$

Find the pinhole size that produces the sharpest image of the light source on the screen.

[4]

$$\begin{aligned} s_3^2 &= s_1^2 + s_2^2 \\ &= x^2 \left(\frac{u+d}{u} \right)^2 + \frac{4\lambda^2 d^2}{x^2} \end{aligned}$$

→ FOR A SHARP IMAGE, MAXIMIZE s_3

DIFFERENTIATE s_3^2 INSTEAD OF s_3 TO SIMPLIFY THE ALGEBRA

$$0 = \frac{ds_3^2}{dx} = 2x \cdot \left(\frac{u+d}{u} \right)^2 - \frac{2}{x^3} \cdot 4\lambda^2 d^2$$

$$\Rightarrow 2x^4 \left(\frac{u+d}{u} \right)^2 = 2 \cdot 4\lambda^2 d^2 \quad / \sqrt{}$$

$$x^2 \frac{u+d}{u} = 2\lambda d$$

$$x = \sqrt{\frac{2\lambda d u}{u+d}}$$

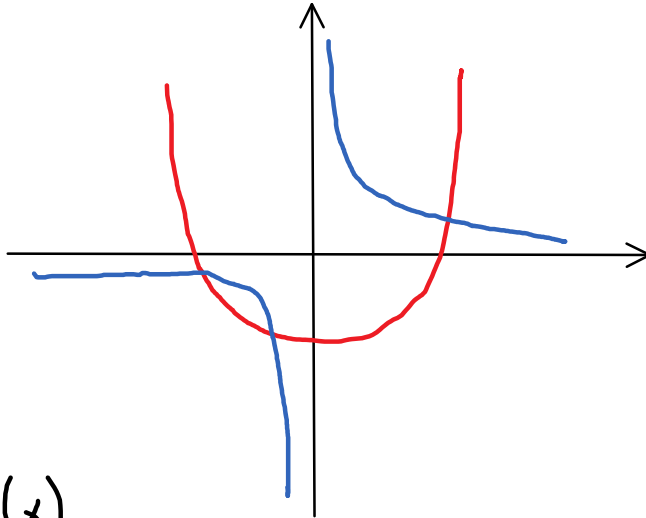
22.

- (a) Sketch the functions $f(x) = x^2 + 2x - 7$ and $g(x) = \frac{2}{x}$ on the same graph. [2]
- (b) Determine the coordinates of the points of intersection of the two curves and label their exact values. You are **not** required to determine coordinates of intersections with x or y axes. [4]
- (c) Evaluate the finite area enclosed between the two curves. [4]

You are expected to show clear working for all parts.

$f(x) = \text{PARABOLIC}$

$g(x) = \text{RECIPROCAL}$



b) $f(x) = g(x)$

$$x^2 + 2x - 7 = \frac{2}{x}$$

$$x^3 + 2x^2 - 7x - 2 = 0 \rightarrow \text{NEEDS TO BE FACTORIZED}$$

$$x^3 + 2x^2 - 8x + x - 2 = 0$$

DO TRIAL & ERROR
TILL YOU GET IT

$$x^3 - 2x^2 + 4x^2 - 8x + x - 2 = 0$$

$$x^2(x-2) + 4x(x-2) + 1 \cdot (x-2) = 0$$

$$(x^2 + 4x + 1)(x-2) = 0$$

i) $x^2 + 4x + 1 = 0$

$$x = \frac{-4 \pm \sqrt{16-4}}{2} = -2 \pm \sqrt{3}$$

ii) $x-2 = 0$

$$x = 2$$

EASIER TO PLUG x INTO $g(x)$ TO FIND y

$$x=2 \quad y = g(2) = \frac{2}{2} = 1 \Rightarrow (2, 1)$$

$$x = \sqrt{3} - 2 \quad y = \frac{2}{\sqrt{3}-2} \rightarrow \text{NEEDS TO BE RATIONALIZED}$$

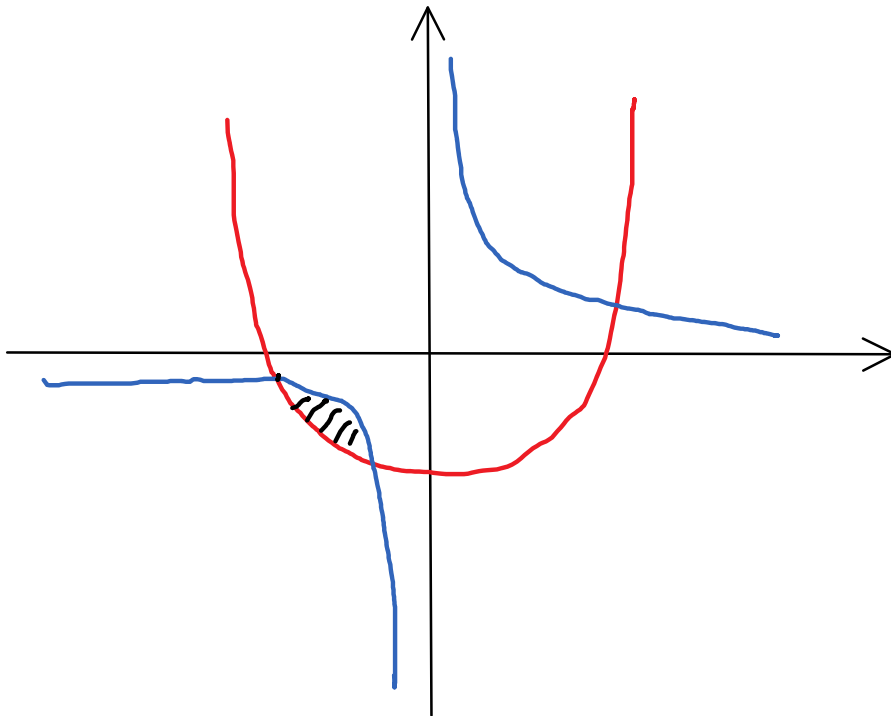
$$y = \frac{2}{\sqrt{3}-2} \cdot \frac{\sqrt{3}+2}{\sqrt{3}+2} = \frac{2\sqrt{3}+4}{3-4} = -2\sqrt{3}-4$$

$$\Rightarrow (\sqrt{3}-2, -2\sqrt{3}-4)$$

$$x = -\sqrt{3}-2 \quad y = \frac{2}{-\sqrt{3}-2} \cdot \frac{-\sqrt{3}+2}{-\sqrt{3}+2} = \frac{-2\sqrt{3}+4}{3-4} = 2\sqrt{3}-4$$

$$\Rightarrow (-\sqrt{3}-2, 2\sqrt{3}-4)$$

c)



$$\text{AREA} = \int_{-\sqrt{3}-2}^{\sqrt{3}-2} f(x) dx - \int_{-\sqrt{3}-2}^{\sqrt{3}-2} g(x) dx$$

$$\text{AREA} = \int_{-\sqrt{3}-2}^{\sqrt{3}-2} x^2 + 2x - 7 - \frac{2}{x} dx$$

$$= \left[\frac{1}{3}x^3 + x^2 - 7x - 2\ln x \right]_{-\sqrt{3}-2}^{\sqrt{3}-2}$$

USE $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

1st TERM: $\frac{1}{3} \left((\sqrt{3}-2)^3 - (-\sqrt{3}-2)^3 \right) =$
 $= \frac{1}{3} (3\sqrt{3} - 18 + 12\sqrt{3} - 8 - (-3\sqrt{3} - 18 - 12 - 8)) = 10\sqrt{3}$

2nd TERM: $(\sqrt{3}-2)^2 - (-\sqrt{3}-2)^2$
 $= 3 - 4\sqrt{3} + 4 - (3 + 4\sqrt{3} + 4) = -8\sqrt{3}$

3rd TERM: $7 \cdot \sqrt{3} - 14 - (-7\sqrt{3} - 14) = 14\sqrt{3}$

USE $\ln a - \ln b = \ln \frac{a}{b}$

4th TERM: $2 \ln(\sqrt{3}-2) - 2 \ln(-\sqrt{3}-2) = 2 \ln \frac{\sqrt{3}-2}{-\sqrt{3}-2}$
 $= 2 \ln \left(\frac{\sqrt{3}-2}{-\sqrt{3}-2} \cdot \frac{-\sqrt{3}+2}{-\sqrt{3}+2} \right) = 2 \ln \left(\frac{-(\sqrt{3}-2)^2}{-1} \right) = 4 \ln(\sqrt{3}-2)$

AREA = $10\sqrt{3} - 8\sqrt{3} - 14\sqrt{3} - 4 \ln(\sqrt{3}-2)$
 $= -12\sqrt{3} - 4 \ln(\sqrt{3}-2)$

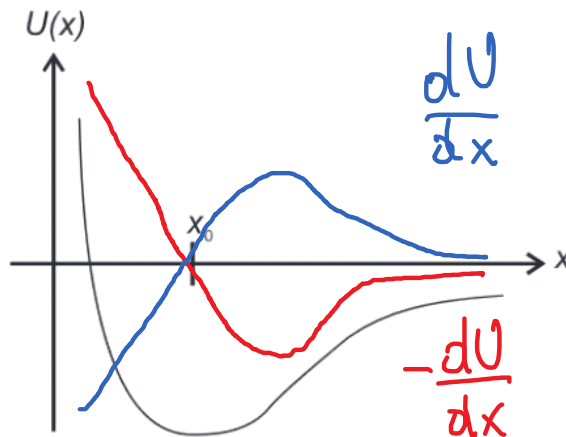
23. The potential energy for a system of two atoms separated by a distance x is $U(x)$ (shown below).

Sketch $-\frac{dU}{dx}$.

[2]

What does this new curve represent physically and what does the point x_0 represent?

[2]



$+\frac{dU}{dx}$ IS THE SLOPE OF U

$x < x_0$ SLOPE NEGATIVE

$x = x_0$ SLOPE ZERO

$x > x_0$ SLOPE POSITIVE

$x \rightarrow \infty$ SLOPE $\rightarrow 0$

$$\text{WORK} = W = \int F dx$$

WHEN YOU DO WORK, YOU LOSE YOUR POTENTIAL ENERGY

THEREFORE $\Delta U = -W$

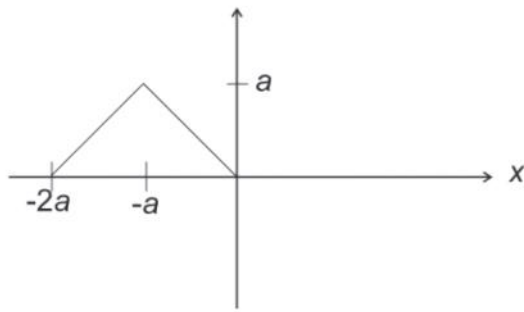
$$\Delta U = - \int F dx$$

$$\frac{dU}{dx} = -F \Rightarrow F = -\frac{dU}{dx} = \text{FORCE}$$

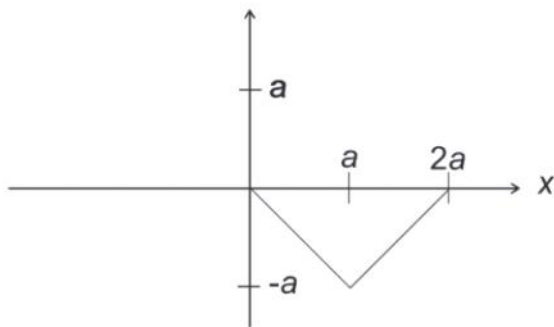
x_0 IS THE POSITION OF ENERGY MINIMUM

$\rightarrow$ EQUILIBRIUM POSITION WHERE $F = 0$

24. If $f(x)$ is



and $g(x)$ is



HORIZONTAL SHIFT

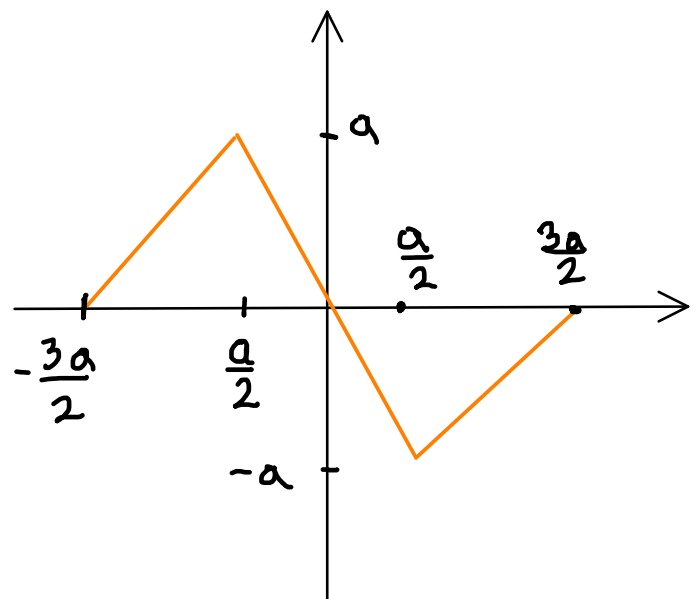
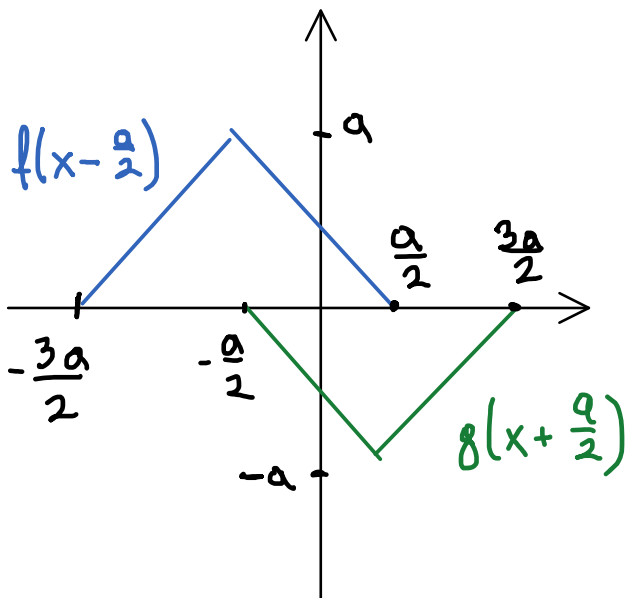
REMEMBER: IT IS COUNTERINTUITIVE
POSITIVE SHIFT IS TO THE LEFT
NEG. SHIFT IS TO THE RIGHT

then sketch $f(x - ct) + g(x + ct)$ for $t = a/2c$, $3a/4c$ and a/c . Each sketch should be done separately with its own set of axes.

[6]

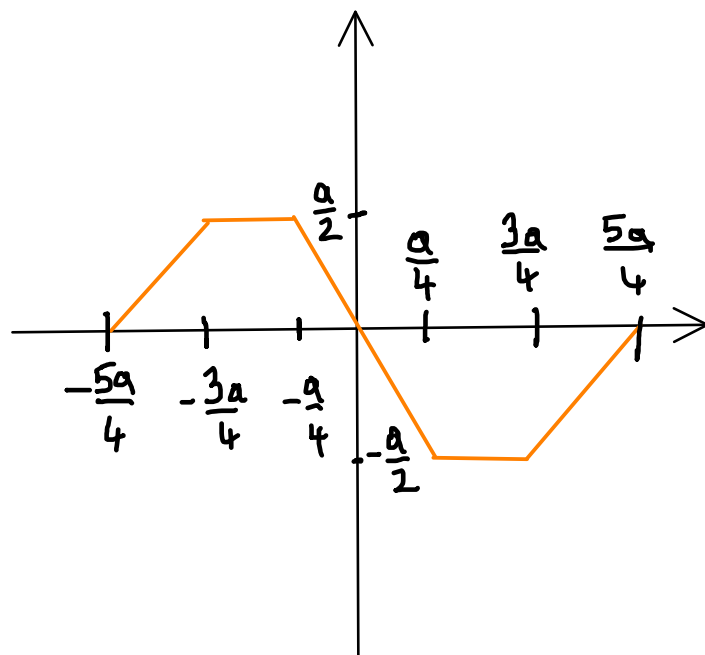
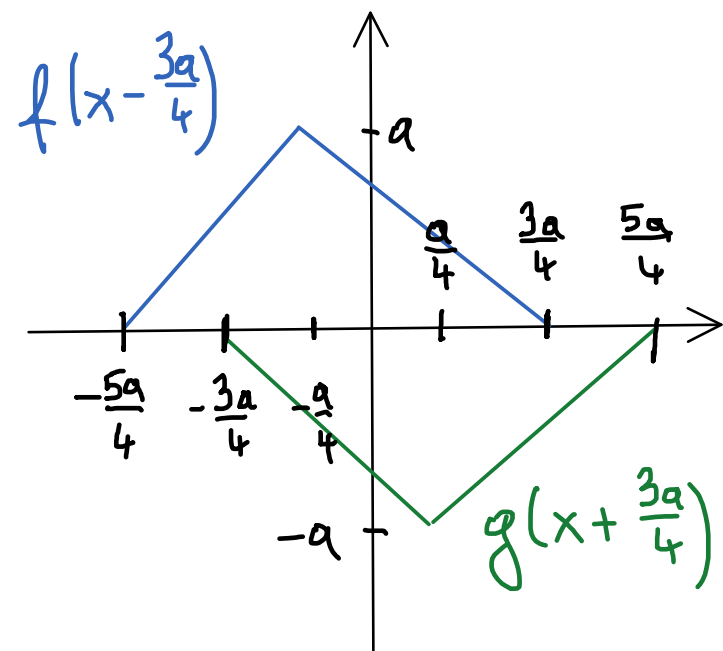
$$t = \frac{a}{2c}$$

$$f\left(x - \frac{a}{2}\right) + g\left(x + \frac{a}{2}\right)$$



$$t = \frac{3a}{4}$$

$$f(x - \frac{3}{4}a) + g(x + \frac{3}{4}a)$$



$$t = a$$

$$f(x - a) + g(x + a)$$

